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LAPLACE TRANSFORMS FOR THE FIRST HITTING TIME OF A BROWNIAN MOTION

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Abstract

The purpose of this paper is to present some propositions about the Laplace transform related to the first hitting time to piecewise linear functions of a Brownian motion. We introduce also a terminal date and examine the lower between the hitting time and the terminal moment. Regardless of whether the hitting time is before the terminal date or not, we shall only know either the stopping time value or the value of the Brownian motion. This requires the separate examination of both cases.

The derived results can be used for pricing financial derivatives related to reaching some boundaries of the underlying asset – for example, barrier or American options.

Key words: Brownian motion, stopping times, first hitting, Laplace transform

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1. Introduction. The problem of first hitting of a Brownian motion to some boundary is examined in many studies – see for example [1], [2], and [3]. In the present article we investigate the Laplace transform of the truncated first hitting time to some piecewise linear function. The truncation is done at some previously defined terminal value. There are many reasons that motivate us to do this research. First, the Laplace transform is just the moment generating function of the stopping time and therefore it is a powerful tool for examining

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its distribution, moments, and cumulants. Second, if we approximate a curve by some piecewise linear functions, we can numerically derive the corresponding results for first hitting to this curve. Third, the wide use of the Brownian motion in different real life areas implies the practical significance of the derived results.

For instance, if a financial market is described by the famous Black–Scholes framework, the asset price is presented by a geometric Brownian motion. Thus the Laplace transform appears in many path dependent financial derivatives. If the hitting time is before the terminal date, then we know the value of the Brownian motion and hence we have to derive the Laplace transform of the hitting time. Otherwise, if the hitting time is after the terminal moment, then the stopping time is equal to the terminal value. Thus we have to derive the Laplace transform of the Brownian motion provided that it does not hit the barrier before the terminal moment.

An outstanding example is the class of the barrier derivatives. Their main feature is that they change their behaviour after reaching some boundary. Other important examples are the American style derivatives. They lead to optimal stopping problems, for which we have to obtain the fair price as well as the optimal stopping boundary. In [4] is presented an algorithm for deriving the early exercise boundary of an American put. A third class of such derivatives are the so-called defaultables. It is assumed that the default occurs if the underlying asset falls below some curve. Pricing of such derivatives is examined in [5] and [6]. Once we know the corresponding boundaries and Laplace transforms we can use different numerical techniques to derive the derivative prices. Some software methods and Monte Carlo simulations for different style options – European, American, basket, and others – are presented in [7], [8], and [9].

The paper is organized as follows. In Section 2 we present the basic notations which we shall use later. In Section 3 we examine the linear boundary problem, whereas in Section 4 are presented the results when the boundary is piecewise linear.

2. Preliminaries. Let B_t be a Brownian motion and T be the terminal moment, $T \leq \infty$. Let $0 \equiv t_0 < t_1 < \cdots < t_n \equiv T$ be the time grid and $b\left(t\right)$ be a piecewise linear function on it

$$b\left(t\right) = \sum_{i=1}^{n} \left(b_{1,i}t + b_{2,i}\right) I_{t\in[t_{i-1},t_i]}, \tag{2.1}$$

where I is the indicator function. Note that it is sufficient to examine upper version of the problem due to the symmetry of the Brownian motion. Let $\beta_i = b\left(t_i\right)$ and therefore $\beta_0 > 0$. We shall assume also that function (2.1) is continuous ($b_i\left(t_i\right) = b_{i-1}\left(t_i\right)$) but the presented results can be easily generalized if it has discontinuities. We shall denote by τ the first hitting time to the function $b\left(\cdot\right)$. Let us notate by Λ_t the indicator process $\Lambda_t = I_{\tau\leq T}$ and by $N\left(\cdot\right)$ the cumulative distribution function of the standard normal distribution.

3. Linear case. Let us suppose that the function $b(t)$ is linear. Note that $b_2 \equiv \beta_0 > 0$. We shall use several times the following result, which is reported in [10], equation (2.0.2) on page 223.

Lemma 3.1. *The probability density function of τ is*

$$p(t) = \frac{b_2}{\sqrt{2\pi} t^{\frac{3}{2}}} \exp\left(-\frac{(b_1 t + b_2)^2}{2t}\right). \tag{3.1}$$

An immediate corollary is the form of the cumulative distribution function.

Proposition 3.1. *The cumulative distribution function of τ, which we shall notate by $g(\cdot)$, is given by the equation*

$$g(T; b_1, b_2) \equiv P(\tau < T) = 1 - N\left(\frac{b_1 T + b_2}{\sqrt{T}}\right) + \exp(-2b_1 b_2) N\left(\frac{b_1 T - b_2}{\sqrt{T}}\right). \tag{3.2}$$

Proof. Since $b_2 > 0$, we have that $g(0; b_1, b_2) = 0$. To finish the proof we check that the derivative $g_t(t; b_1, b_2)$ is given by density (3.1). $\square$

Our first result is established in the following theorem.

Theorem 3.1. *Let $\theta > 0$. The Laplace transform of τ before T is given by*

$$L(T, \theta; b_1, b_2) = E\left[e^{-\theta\tau} \Lambda_T\right] = e^{b_2\left(\sqrt{b_1^2 + 2\theta} - b_1\right)} g\left(T; \sqrt{b_1^2 + 2\theta}, b_2\right), \tag{3.3}$$

where the function $g(\cdot)$ is given by equation (3.2).

Proof. Using Lemma 3.1 and Proposition 3.1 we derive

$$\begin{aligned} E\left[e^{-\theta\tau} \Lambda_T\right] &= \int_0^\infty e^{-\theta t} I_{t \le T} \frac{b_2}{\sqrt{2\pi} t^{\frac{3}{2}}} \exp\left(-\frac{(b_1 t + b_2)^2}{2t}\right) dt \\ &= e^{b_2\left(\sqrt{b_1^2 + 2\theta} - b_1\right)} \int_0^T \frac{b_2}{\sqrt{2\pi} t^{\frac{3}{2}}} \exp\left(-\frac{\left(\sqrt{b_1^2 + 2\theta}\, t + b_2\right)^2}{2t}\right) dt \\ &= e^{b_2\left(\sqrt{b_1^2 + 2\theta} - b_1\right)} g\left(T; \sqrt{b_1^2 + 2\theta}, b_2\right). \end{aligned} \tag{3.4}$$

$\square$

The following corollary gives the results if there is not a terminal moment.

Corollary 3.1. *Suppose that $T = \infty$. The probability τ to be finite is*

$$g(\infty; b_1, b_2) \equiv P(\tau < \infty) = \begin{cases} 1 & if \quad b_1 \le 0 \\ \exp(-2b_1 b_2) & if \quad b_1 > 0 \end{cases}.$$

The corresponding Laplace transform is given by

$$L(\infty, \theta; b_1, b_2) = E\left[e^{-\theta\tau} I_{\tau < \infty}\right] = e^{-b_2\left(\sqrt{b_1^2 + 2\theta} + b_1\right)}.$$

Now we turn to the case when the Brownian motion does not hit the linear function before the moment T. We shall use the following lemma whose proof can be found in [11].

Lemma 3.2. *If $z < b(T)$, then the probability of $\tau > T$, conditioned on $B_T = z$ is*

$$P(\tau > T |\, B_T = z) = 1 - \exp\left(-\frac{2b_2\,(b(T) - z)}{T}\right). \tag{3.5}$$

Now we can prove our second result.

Theorem 3.2. *If $z < b(T)$, then*

$$\begin{aligned} &V(\theta, z, T; b_1, b_2) \equiv E\left[e^{\theta B_T} I_{B_T > z, \tau > T}\right] = \\ &= \exp\left(\frac{T\theta^2}{2}\right)\left[\begin{array}{l} N\left(\dfrac{b(T) - T\theta}{\sqrt{T}}\right) - N\left(\dfrac{z - T\theta}{\sqrt{T}}\right) \\ +e^{2b_2(\theta - b_1)}\left(N\left(\dfrac{z - T\theta - 2b_2}{\sqrt{T}}\right) - N\left(\dfrac{b(T) - T\theta - 2b_2}{\sqrt{T}}\right)\right)\end{array}\right]. \end{aligned} \tag{3.6}$$

Proof. Using Lemma 3.2 we derive

$$\begin{aligned} P(B_T < y, \tau > T) &= \frac{1}{\sqrt{2\pi T}}\int_{-\infty}^{\infty} P(B_T < y, \tau > T |\, B_T = u)\exp\left(-\frac{u^2}{2T}\right) du \\ &= \frac{1}{\sqrt{2\pi T}}\int_{-\infty}^{y}\left(1 - \exp\left(-\frac{2b_2\,(b(T) - u)}{T}\right)\right)\exp\left(-\frac{u^2}{2T}\right) du. \end{aligned} \tag{3.7}$$

Having in mind that $P(B_T < y, \tau > T) = P(B_T < b(T), \tau > T)$ when $y > b(T)$, we obtain

$$\begin{aligned} E\left[e^{\theta B_T} I_{B_T > z, \tau > T}\right] &= \int_{z}^{b(T)} e^{\theta y} dP(B_T < y, \tau > T) \\ &= \frac{1}{\sqrt{2\pi T}}\int_{z}^{b(T)} e^{\theta y}\left(1 - \exp\left(-\frac{2b_2\,(b(T) - y)}{T}\right)\right)\exp\left(-\frac{y^2}{2T}\right) dy. \end{aligned} \tag{3.8}$$

Dividing integral (3.8) into two parts we derive formula (3.6). □

4. Piecewise linear case. The density of the stopping time is given in the following lemma, whose proof can be found in [1] or [3].

Lemma 4.1. *Let t belong to the m-th interval, $t_{m-1} < t < t_m$. Then the τ-density in the point t is*

$$\int_{-\infty}^{\beta_1,\ldots,\beta_{m-1}} \left(\begin{array}{l} \prod_{i=1}^{m-1} \left(1 - \exp\left(-\frac{2(\beta_{i-1}-x_{i-1})(\beta_i - x_i)}{t_i - t_{i-1}}\right)\right) \\ \prod_{i=1}^{m-1} \frac{\exp\left(-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}\right)}{\sqrt{2\pi(t_i - t_{i-1})}} \\ \frac{\beta_{m-1}-x_{m-1}}{\sqrt{2\pi}(t-t_{m-1})^{\frac{3}{2}}} \exp\left(-\frac{(b_{1,m}t + b_{2,m} - x_{m-1})^2}{2(t-t_{m-1})}\right) \end{array} \right) dx_1 \cdots dx_{m-1}. \tag{4.1}$$

The following theorem corresponds to Theorem 3.1 when the boundary is piecewise linear.

Theorem 4.1. *Let $\theta > 0$. The Laplace transform of the first hitting time in the m-th interval is given by*

$$\begin{aligned} & E\left[e^{-\theta\tau} I_{\tau \in (t_{m-1}, t_m]}\right] \\ & = \int_{-\infty}^{\beta_1,\ldots,\beta_{m-1}} \left(\begin{array}{l} \prod_{i=1}^{m-1} \left(1 - \exp\left(-\frac{2(\beta_{i-1}-x_{i-1})(\beta_i - x_i)}{t_i - t_{i-1}}\right)\right) \\ \prod_{i=1}^{m-1} \frac{\left(\exp\left(-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}\right)\right)}{\sqrt{2\pi(t_i - t_{i-1})}} \\ e^{-\theta t_{m-1}} L\left(t_m - t_{m-1}, \theta; b_{1,m}, \beta_{m-1} - x_{m-1}\right) \end{array} \right) dx_1 \cdots dx_{m-1}, \end{aligned} \tag{4.2}$$

where the function $L(\cdot)$ is given by equation (3.3).

Proof. Using the form of stopping time density (4.1) we derive

$$\begin{aligned} & E\left[e^{-\theta\tau} I_{\tau \in (t_{m-1}, t_m]}\right] \\ & = \int_{t_{m-1}}^{t_m} \int_{-\infty}^{\beta_1,\ldots,\beta_{m-1}} \left(\begin{array}{l} \prod_{i=1}^{m-1} \left(1 - \exp\left(-\frac{2(\beta_{i-1}-x_{i-1})(\beta_i - x_i)}{t_i - t_{i-1}}\right)\right) \\ \prod_{i=1}^{m-1} \frac{\exp\left(-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}\right)}{\sqrt{2\pi(t_i - t_{i-1})}} \\ e^{-\theta t} \frac{\beta_{m-1}-x_{m-1}}{\sqrt{2\pi}(t-t_{m-1})^{\frac{3}{2}}} \exp\left(-\frac{(b_{1,m}t + b_{2,m} - x_{m-1})^2}{2(t-t_{m-1})}\right) \end{array} \right) dx_1 \cdots dx_{m-1} dt \\ & = \int_{-\infty}^{\beta_1,\ldots,\beta_{m-1}} \left(\begin{array}{l} \prod_{i=1}^{m-1} \left(1 - \exp\left(-\frac{2(\beta_{i-1}-x_{i-1})(\beta_i - x_i)}{t_i - t_{i-1}}\right)\right) \\ \prod_{i=1}^{m-1} \frac{\exp\left(-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}\right)}{\sqrt{2\pi(t_i - t_{i-1})}} \\ e^{-\theta t_{m-1}} \int_0^{t_m - t_{m-1}} \left(\begin{array}{l} e^{-\theta u} \frac{\beta_{m-1}-x_{m-1}}{\sqrt{2\pi} u^{\frac{3}{2}}} \\ \exp\left(-\frac{(b_{1,m}u + \beta_{m-1} - x_{m-1})^2}{2u}\right) \end{array} \right) du \end{array} \right) dx_1 \cdots dx_{m-1}. \end{aligned} \tag{4.3}$$

We have changed the variable t to $u = t - t_{m-1}$ above using $\beta_{m-1} = b_{1,m} t_{m-1} + b_{2,m}$. We finish the proof using Theorem 3.1. $\square$

The corresponding to Theorem 3.2 result in the piecewise linear case is as follows.

Theorem 4.2. *If $z < b(T)$, then the corresponding Laplace transform when first hitting is after the terminal moment is*

$$\text{(4.4)}\quad E\left[e^{\theta B_T} I_{B_T>z,\tau>T}\right] = \int_{-\infty}^{\beta_1,\dots,\beta_{n-1}} \begin{pmatrix} \prod_{i=1}^{n-1}\left(1-\exp\left(-\frac{2(\beta_{i-1}-x_{i-1})(\beta_i-x_i)}{t_i-t_{i-1}}\right)\right) \\ \prod_{i=1}^{n-1}\frac{\left(\exp\left(-\frac{(x_i-x_{i-1})^2}{2(t_i-t_{i-1})}\right)\right)}{\sqrt{2\pi(t_i-t_{i-1})}} \\ e^{\theta x_{n-1}} V\left(\theta, z-x_{n-1}, t_n-t_{n-1}; b_{1,n-1}, b_{2,n-1}-x_{n-1}\right) \end{pmatrix} dx_1\cdots dx_{n-1},$$

where the function $V(\cdot)$ is given by equation (3.6).

Proof. Note again that $P(B_T < y, \tau > T) = P(B_T < b(T), \tau > T)$ when $y > b(T)$. Using the Markovian property of the Brownian motion we derive

$$\text{(4.5)}\quad E\left[e^{\theta B_T} I_{B_T>z,\tau>T}\right] = \int_z^{b_{n(T)}} e^{\theta u} dP\left(B_T < u, \tau > T\right)$$

$$= \int_z^{b_{n(T)}} e^{\theta u} d\left(\int_{-\infty}^{\beta_1,\dots,\beta_{n-1}} \begin{pmatrix} P\left(B_T < u, \tau > T \mid B_{t_1} = x_1, \dots, B_{t_{n-1}} = x_{n-1}\right) \\ \prod_{i=1}^{n-1}\frac{\left(\exp\left(-\frac{(x_i-x_{i-1})^2}{2(t_i-t_{i-1})}\right)\right)}{\sqrt{2\pi\left(t_i-t_{i-1}\right)}} \end{pmatrix} dx_1\cdots dx_{n-1}\right)$$

$$= \int_z^{b_{n(T)}} e^{\theta u} d\left(\int_{-\infty}^{\beta_1,\dots,\beta_{n-1}} \begin{pmatrix} \prod_{i=1}^{n-1} P^{t_{i-1},x_{i-1}}\left(\tau > t_i \mid B_{t_i} = x_i\right) \\ \prod_{i=1}^{n-1}\frac{\left(\exp\left(-\frac{(x_i-x_{i-1})^2}{2(t_i-t_{i-1})}\right)\right)}{\sqrt{2\pi\left(t_i-t_{i-1}\right)}} \\ P^{t_{n-1},x_{n-1}}\left(\tau > T, B_T < u\right) \end{pmatrix} dx_1\cdots dx_{n-1}\right)$$

$$= \int_{-\infty}^{\beta_1,\dots,\beta_{n-1}} \begin{pmatrix} \prod_{i=1}^{n-1} P^{t_{i-1},x_{i-1}}\left(\tau > t_i \mid B_{t_i} = x_i\right) \\ \prod_{i=1}^{n-1}\frac{\left(\exp\left(-\frac{(x_i-x_{i-1})^2}{2(t_i-t_{i-1})}\right)\right)}{\sqrt{2\pi\left(t_i-t_{i-1}\right)}} \\ \int_z^{b_{n(T)}} e^{\theta u} dP^{t_{n-1},x_{n-1}}\left(\tau > T, B_T < u\right) \end{pmatrix} dx_1\cdots dx_{n-1}.$$

We change the variable u to $v = u - x_{n-1}$ and apply Theorem 3.2 to transform the inner integral to

$$\int\limits_{z}^{b_{n(T)}} e^{\theta u} dP^{t_{n-1},x_{n-1}} \left(\tau > T, B_T < u\right) = \int\limits_{z}^{b_{n(T)}} e^{\theta u} dP \left(\tau > T - t_{n-1}, B_T < u - x_{n-1}\right)$$

$$= e^{\theta x_{n-1}} \int\limits_{z-x_{n-1}}^{b_{n(T)}-x_{n-1}} e^{\theta v} dP \left(\tau > T - t_{n-1}, B_T < v\right)$$

$$= e^{\theta x_{n-1}} V \left(\theta, z - x_{n-1}, t_n - t_{n-1}; b_{1,n-1}, b_{2,n-1} - x_{n-1}\right).$$

Using Lemma 3.2 we finish the proof. □

5. Conclusions and further work. We derived the Laplace transforms related to the first hitting time to a piecewise linear function of a Brownian motion. We introduced a terminal date and examined separately the cases when hitting occurs before or after this moment. The derived Laplace transforms are presented in Theorems 3.1 and 3.2 when the boundary is linear, whereas Theorems 4.1 and 4.2 concern the piecewise linear case. We can apply these results to price some path dependent securities. Important examples are barrier and American style derivatives as well as defaultables.

An interesting open problem is deriving the corresponding Laplace transforms when the stopping time is the first exit from a piecewise linear strip. These results would be useful if we have to price double barrier options or game style options.

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